Quantile-Based Nonparametric Inference for First-Price Auctions

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Abstract

We propose a quantile-based nonparametric approach to inference on the probability density function (PDF) of the private values in first-price sealed-bid auctions with independent private values. Our method of inference is based on a fully nonparametric kernel-based estimator of the quantiles and PDF of observable bids. Our estimator attains the optimal rate of Guerre, Perrigne, and Vuong (2000), and is also asymptotically normal with the appropriate choice of the bandwidth. As an application, we consider the problem of inference on the optimal reserve price.

Keywords: First-price auctions, independent private values, nonparametric estimation, kernel estimation, quantiles, optimal reserve price.

1 Introduction

Following the seminal article of Guerre, Perrigne, and Vuong (2000), GPV hereafter, there has been an enormous interest in nonparametric approaches to auctions.[1] By removing the need to impose tight functional form assumptions, the nonparametric approach provides a more flexible framework for estimation and inference. Moreover,

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[1]See a recent survey by Athey and Haile (2005).

the sample sizes available for auction data can be sufficiently large to make the non-parametric approach empirically feasible.[2] This paper contributes to this literature by providing a fully nonparametric framework for making inferences on the density of bidders' valuations $f(v)$. The need to estimate the density of valuations arises in a number of economic applications, as for example the problem of estimating a revenue-maximizing reserve price.[3]

As a starting point, we briefly discuss the estimator proposed in GPV. For the purpose of introduction, we adopt a simplified framework. Consider a random, i.i.d. sample b_{il} of bids in first-price auctions each of which has n bidders; l indexes auctions and $i = 1, \ldots, n$ indexes bids in a given auction. GPV assume independent private values (IPV). In equilibrium, the bids are related to the valuations via the equilibrium bidding strategy B: $b_{il} = B(v_{il})$. GPV show that the inverse bidding strategy is identified directly from the observed distribution of bids:

$$v = \xi(b) \equiv b + \frac{1}{n-1} \frac{G(b)}{g(b)}, \tag{1}$$

where $G(b)$ is the cumulative distribution function (CDF) of bids in an auction with n bidders, and $g(b)$ is the corresponding density. GPV propose to use nonparametric estimators $\hat{G}$ and $\hat{g}$. When $b = b_{il}$, the left-hand side of (1) will then give what GPV call the pseudo-values $\hat{v}_{il} = \hat{\xi}(b_{il})$. The CDF $F(v)$ is estimated as the empirical CDF, and the PDF $f(v)$ is estimated by the method of kernels, both using $\hat{v}_{il}$ as observations. GPV show that, with the appropriate choice of the bandwidth, their estimator converges to the true value at the optimal rate (in the minimax sense; Khasminskii (1978)). However, the asymptotic distribution of this estimator is as yet unknown, possibly because both steps of the GPV method are nonparametric with estimated values $\hat{v}_{il}$ entering the second stage, and because the GPV estimator

[2]For example, List, Daniel, and Michael (2004) study bidder collusion in timber auctions using thousands of auctions conducted in the Province of British Columbia, Canada. Samples of similar size are also available for highway procurement auctions in the United States (e.g., Krasnokutskaya (2003)).

[3]This is an important real-world problem that arises in the administration of timber auctions, for example. The actual objectives of the agencies that auction timber may vary from country to country. In the United States, obtaining a fair price is the main objective of the Forest Service. As observed in Haile and Tamer (2003), this is a vague objective, and determining the revenue maximizing reserve price should be part of the cost-benefits analysis of the Forest Service's policy. In other countries, maximizing the expected revenue from each and every auction is a stated objective, as is for example the case for BC Timber Sales (Roise, 2005).

requires trimming.

The estimator $\hat{f}(v)$ proposed in this paper avoids the use of pseudo-values and does not involve trimming; it builds instead on the insight of Haile, Hong, and Shum (2003).[4] They show that the quantiles of the distribution of valuations can be expressed in terms of the quantiles, PDF, and CDF of bids. We show below that this relation can be used for estimation of $f(v)$. Consider the τ-th quantile of valuations $Q(\tau)$ and the τ-th quantile of bids $q(\tau)$. The latter can be easily estimated from the sample by a variety of methods available in the literature. As for the quantile of valuations, since the inverse bidding strategy $\xi(b)$ is monotone, equation (1) implies that $Q(\tau)$ is related to $q(\tau)$ as follows:

$$Q(\tau) = q(\tau) + \frac{\tau}{(n-1)\, g(q(\tau))}, \tag{2}$$

providing a way to estimate $Q(\tau)$ by a plug-in method. The CDF $F(v)$ can then be recovered simply by inverting the quantile function, $F(v) = Q^{-1}(v)$.

Our estimator $\hat{f}(v)$ is based on a simple idea that by differentiating the quantile function we can recover the density: $Q'(\tau) = 1/f(Q(\tau))$, and therefore $f(v) = 1/Q'(F(v))$. Taking the derivative in (2) and using the fact that $q'(\tau) = 1/g(q(\tau))$, we obtain, after some algebra, our basic formula:

$$f(v) = \left(\frac{n}{n-1} \frac{1}{g(q(F(v)))} - \frac{1}{n-1} \frac{F(v)\, g'(q(F(v)))}{g^3(q(F(v)))} \right)^{-1}. \tag{3}$$

Note that all the quantities on the right-hand side, i.e. $g(b)$, $g'(b)$, $q(\tau)$, $F(v) = Q^{-1}(v)$ can be estimated nonparametrically, for example, using kernel-based methods. Once this is done, we can plug them in (3) to obtain our nonparametric estimator.

The expression in (3) can be also derived the relationship between the CDF of values and the CDF of bids:

$$F(v) = G(B(v)).$$

Applying the change of variable argument to the above identity, one obtains

$$f(v) \;=\; g(B(v))\, B'(v)$$

[4] The focus of Haile, Hong, and Shum (2003) is a test of common values. Their model is therefore different from the IPV model, and requires an estimator that is different from the one in GPV. See also Li, Perrigne, and Vuong (2002).

$$
\begin{aligned}
&= g\left(B\left(v\right)\right)/\xi'\left(B\left(v\right)\right) \\
&= \left(\frac{n}{n-1}\frac{1}{g\left(B\left(v\right)\right)}-\frac{1}{n-1}\frac{F\left(v\right)g'\left(B\left(v\right)\right)}{g^3\left(B\left(v\right)\right)}\right)^{-1}.
\end{aligned}
$$

Note however, that, from the estimation perspective, the quantile-based formula appears to be more convenient, since the bidding strategy function B involves integration of F (see GPV). Furthermore, as we show below, quantile-based approach eliminates trimming, which is likely to be one of the factors preventing one from establishing asymptotic normality of the GPV estimator.

Our framework results in the estimator of $f(v)$ that is both consistent and asymptotically normal, with an asymptotic variance that can be easily estimated. Moreover, we show that, with an appropriate choice of the bandwidth sequence, the proposed estimator attains the minimax rate of GPV.

As an application, we consider the problem of inference on the optimal reserve price. Several previous articles have considered the problem of estimating the optimal reserve price. Paarsch (1997) develops a parametric approach and applies his estimator to timber auctions in British Columbia. Haile and Tamer (2003) consider the problem of inference in an incomplete model of English auction, derive nonparametric bounds on the reserve price and apply them to the reserve price policy in the US Forest Service auctions. Closer to the subject of our paper, Li, Perrigne, and Vuong (2003) develop a semiparametric method to estimate the optimal reserve price. At a simplified level, their method essentially amounts to re-formulating the problem as a maximum estimator of the seller's expected profit. Strong consistency of the estimator is shown, but its asymptotic distribution is as yet unknown.

In this paper, we propose asymptotic confidence intervals (CIs) for the optimal reserve price. Our CIs are formed by inverting a collection of asymptotic tests of Riley and Samuelson's (1981) equation determining the optimal reserve price. This equation involves the density $f(v)$, and a test statistic with an asymptotically normal distribution under the null can be constructed using our estimator.

The paper is organized as follows. Section 2 introduces the basic setup. Similarly to GPV, we allow the number of bidders to vary from auctions to auction, and also allow auction-specific covariates. Section 3 presents our main results. Section 4 discusses inference on the optimal reserve price. We report Monte Carlo results in Section 5. Section 6 concludes. All proofs are contained in the Appendix.

2 Definitions

Suppose that the econometrician observes the random sample $\{(b_{il}, x_l, n_l) : l = 1, \ldots, L; i = 1, \ldots n_l\}$, where b_{il} is an equilibrium bid of bidder i submitted in auction l with n_l bidders, and x_l is the vector of auction-specific covariates for auction l. The corresponding unobservable valuations of the object are given by $\{v_{il} : l = 1, \ldots, L; i = 1, \ldots n_l\}$. We make the following assumption about the data generating process.

Assumption 1 **(a)** $\{(n_l, x_l) : l = 1, \ldots, L\}$ *are i.i.d.*

(b) *The marginal PDF of x_l, φ, is strictly positive and continuous on its compact support $\mathcal{X} \subset R^d$, and admits at least $R \geq 2$ continuous derivatives on its interior.*

(c) *The distribution of n_l conditional on x_l is denoted by $\pi(n|x)$ and has support $\mathcal{N} = \{\underline{n}, \ldots, \bar{n}\}$ for all $x \in \mathcal{X}$, $\underline{n} \geq 2$.*

(d) *$\{v_{il} : i = 1, \ldots n; l = 1, \ldots, L\}$ are i.i.d. conditional on x_l with the PDF $f(v|x)$ and CDF $F(v|x)$.*

(e) *$f(\cdot|\cdot)$ is strictly positive and bounded away from zero on its support, a compact interval $[\underline{v}(x), \overline{v}(x)] \subset \mathbb{R}_+$, and admits at least R continuous partial derivatives on $\{(v, x) : v \in (\underline{v}(x), \overline{v}(x)), x \in Interior(\mathcal{X})\}$.*

(f) *For all $n \in \mathcal{N}$, $\pi(n|\cdot)$ admits at least R continuous derivatives on the interior of $\mathcal{X}$.*

In the equilibrium and under Assumption 1(c), the equilibrium bids are determined by

$$b_{il} = v_{il} - \frac{1}{(F(v_{il}|x_l))^{n-1}} \int_{\underline{v}}^{v_{il}} (F(u|x_l))^{n-1} du,$$

(see, for example, GPV). Let $g(b|n, x)$ and $G(b|n, x)$ be the PDF and CDF of b_{il}, conditional on both $x_l = x$ and the number of bidders $n_l = n$. Since b_{il} is a function of v_{il}, x_l and $F(\cdot|x_l)$, the bids $\{b_{il}\}$ are also i.i.d. conditional on (n_l, x_l). Furthermore, by Proposition 1(i) and (iv) of GPV, for all $n = \underline{n}, \ldots, \overline{n}$ and $x \in \mathcal{X}$, $g(b|n, x)$ has the compact support $\left[\underline{b}(n, x), \overline{b}(n, x)\right]$ for some $\underline{b}(n, x) < \overline{b}(n, x)$ and admits at least $R + 1$ continuous bounded partial derivatives.

The τ-th quantile of $F(v|x)$ is defined as

$$\begin{aligned} Q(\tau|x) &= F^{-1}(\tau|x) \\ &\equiv \inf_v \{v : F(v|x) \geq \tau\}. \end{aligned}$$

The τ-th quantile of G, $q(\tau|n,x) = G^{-1}(\tau|n,x)$, is defined similarly. The quantiles of the distributions $F(v|x)$ and $G(b|n,x)$ are related through the following conditional version of equation (2):

$$Q(\tau|x) = q(\tau|n,x) + \frac{\tau}{(n-1)\, g(q(\tau|n,x)|n,x)}. \tag{4}$$

Note that the expression on the left-hand side does not depend on n, since, as it is assumed in the literature, the distribution of valuations is the same regardless of the number of bidders.

The true distribution of the valuations is unknown to the econometrician. Our objective is to construct a valid asymptotic inference procedure for the unknown f using the data on observable bids. Differentiating (4) with respect to τ, we obtain the following equation relating the PDF of valuations with functionals of the distribution of the bids:

$$\begin{aligned} \frac{\partial Q(\tau|x)}{\partial \tau} &= \frac{1}{f(Q(\tau|x)|x)} \\ &= \frac{n}{n-1}\frac{1}{g(q(\tau|n,x)|n,x)} - \frac{\tau g^{(1)}(q(\tau|n,x)|n,x)}{(n-1)\, g^3(q(\tau|n,x)|n,x)}, \end{aligned} \tag{5}$$

where $g^{(k)}(b|n,x) = \partial^k g(b|n,x)/\partial b^k$. Substituting $\tau = F(v|x)$ in equation (5) and using the identity $Q(F(v|x)|x) = v$, we obtain the following equation that represents the PDF of valuations in terms of the quantiles, PDF and derivative of PDF of bids:

$$\begin{aligned} \frac{1}{f(v|x)} &= \frac{n}{n-1}\frac{1}{g(q(F(v|x)|n,x)|n,x)} \\ &\quad - \frac{1}{n-1}\frac{F(v|x)\, g^{(1)}(q(F(v|x)|n,x)|n,x)}{g^3(q(F(v|x)|n,x)|n,x)}. \end{aligned} \tag{6}$$

Note that the overidentifying restriction of the model is that $f(v|x)$ is the same for all n.

In this paper, we suggest a nonparametric estimator for the PDF of valuations based on equations (4) and (6). Such an estimator requires nonparametric estimation of the conditional CDF and quantile functions, PDF and its derivative.[5] Let K be a kernel function. We assume that the kernel is compactly supported and of order R.

Assumption 2 *K is compactly supported on $[-1,1]$, has at least R derivatives on $\mathbb{R}$, the derivatives are Lipschitz, and $\int K(u)\,du = 1$, $\int u^k K(u)\,du = 0$ for $k = 1,\ldots,R-1$.*

To save on notation, denote

$$K_h(z) = \frac{1}{h} K\left(\frac{z}{h}\right),$$

and for $x = (x_1, \ldots, x_d)'$, define

$$K_{*h}(x) = \frac{1}{h^d} K_d\left(\frac{x}{h}\right) = \frac{1}{h^d} \prod_{k=1}^{d} K\left(\frac{x_k}{h}\right).$$

Consider the following estimators:

$$\begin{aligned}
\hat{\varphi}(x) &= \frac{1}{L}\sum_{l=1}^{L} K_{*h}(x_l - x), \qquad (7)\\
\hat{\pi}(n|x) &= \frac{1}{\hat{\varphi}(x) L}\sum_{l=1}^{L} 1(n_l = n) K_{*h}(x_l - x),\\
\hat{G}(b|n,x) &= \frac{1}{\hat{\pi}(n|x)\hat{\varphi}(x) nL}\sum_{l=1}^{L}\sum_{i=1}^{n_l} 1(n_l = n) 1(b_{il} \le b) K_{*h}(x_l - x),\\
\hat{q}(\tau|n,x) &= \hat{G}^{-1}(\tau|n,x) \equiv \inf_b \left\{ b : \hat{G}(b|n,x) \ge \tau \right\},\\
\hat{g}(b|n,x) &= \frac{1}{\hat{\pi}(n|x)\hat{\varphi}(x) nL}\\
&\quad \times \sum_{l=1}^{L}\sum_{i=1}^{n_l} 1(n_l = n) K_h(b_{il} - b) K_{*h}(x_l - x), \qquad (8)
\end{aligned}$$

[5]Nonparametric estimation of conditional CDFs and quantile functions received much attention in the recent econometrics literature (see, for example, Matzkin (2003), and Li and Racine (2005)).

where $1(S)$ is an indicator function of a set $S \subset \mathbb{R}$.[6] The derivatives of the density $g(b|n,x)$ are estimated simply by the derivatives of $\hat{g}(b|n,x)$:

$$\hat{g}^{(k)}(b|n,x) = \frac{1}{\hat{\pi}(n|x)\hat{\varphi}(x)nL} \times \sum_{l=1}^{L}\sum_{i=1}^{n_l} 1(n_l = n) K_h^{(k)}(b_{il} - b) K_{*h}(x_l - x), \quad (9)$$

where $K_h^{(k)}(u) = \frac{1}{h^{1+k}} K^{(k)}(u/h)$, $k = 0, \ldots, R$, and $K^{(0)}(u) = K(u)$.

Our approach also requires nonparametric estimation of Q, the conditional quantile function of valuations. An estimator for Q can be constructed using the relationship between Q, q and g given in (4). A similar estimator was proposed by Haile, Hong, and Shum (2003) in a related context. In our case, the estimator of Q will be used to construct $\hat{F}$, an estimator of the conditional CDF of valuations. Since F is related to Q through

$$F(v|x) = Q^{-1}(v|x) = \sup_{\tau \in [0,1]} \{\tau : Q(\tau|x) \leq v\}, \quad (10)$$

$\hat{F}$ can be obtained by inverting the estimator of the conditional quantile function. However, since an estimator of Q based on (4) involves kernel estimation of the PDF g, it will be inconsistent for the values of τ that are close to zero and one. In particular, such an estimator can exhibit large oscillations for τ near one taking on very small values, which, due to supremum in (10), might proliferate and bring an upward bias into the estimator of F. A possible solution to this problem that we pursue in this paper is to use a monotone version of the estimator of Q. First, we define a preliminary estimator, $\hat{Q}^p$:

$$\hat{Q}^p(\tau|n,x) = \hat{q}(\tau|n,x) + \frac{\tau}{(n-1)\hat{g}(\hat{q}(\tau|n,x)|n,x)}. \quad (11)$$

[6] The quantile estimator $\hat{q}$ is constructed by inverting the estimator of the conditional CDF of bids. This approach is similar to that of Matzkin (2003).

Next, pick τ_0 sufficiently far from 0 and 1, for example, $\tau_0 = 1/2$. We define a monotone version of the estimator of Q as follows.

$$\hat{Q}(\tau|n,x) = \begin{cases} \sup_{t\in[\tau_0,\tau]} \hat{Q}^p(t|n,x), & \tau_0 \le \tau < 1, \\ \inf_{t\in[\tau,\tau_0]} \hat{Q}^p(t|n,x), & 0 \le \tau < \tau_0. \end{cases} \tag{12}$$

The estimator of the conditional CDF of the valuations based on $\hat{Q}(\tau|n,x)$ is given by

$$\hat{F}(v|n,x) = \sup_{\tau\in[0,1]} \left\{ \tau : \hat{Q}(\tau|n,x) \le v \right\}. \tag{13}$$

Since $\hat{Q}(\cdot|n,x)$ is monotone, $\hat{F}$ is not affected by $\hat{Q}^p(\tau|n,x)$ taking on small values near $\tau = 1$. Furthermore, in our framework, inconsistency of $\hat{Q}(\tau|n,x)$ near the boundaries does not pose a problem, since we are interested in estimating F only on a compact inner subset of its support.

Using (6), we propose to estimate $f(v|x)$ by the following nonparametric empirical quantiles-based estimator:

$$\hat{f}(v|x) = \sum_{n=\underline{n}}^{\bar{n}} \hat{\pi}(n|x) \hat{f}(v|n,x), \tag{14}$$

where $\hat{f}(v|n,x)$ is estimated by the plug-in method, i.e. by replacing $g(b|n,x)$, $q(\tau|n,x)$ and $F(v|x)$ in (6) with $\hat{g}(b|n,x)$, $\hat{q}(\tau|n,x)$ and $\hat{F}(v|n,x)$. That is $\hat{f}(v|n,x)$ is given by the reciprocal of

$$\begin{aligned} &\frac{n}{n-1} \frac{1}{\hat{g}\left(\hat{q}\left(\hat{F}(v|n,x)|n,x\right)|n,x\right)} \\ &-\frac{1}{n-1} \frac{\hat{F}(v|n,x)\hat{g}^{(1)}\left(\hat{q}\left(\hat{F}(v|n,x)|n,x\right)|n,x\right)}{\hat{g}^3\left(\hat{q}\left(\hat{F}(v|n,x)|n,x\right)|n,x\right)}. \end{aligned} \tag{15}$$

We also suggest to estimate the conditional CDF of v using the average of $\hat{F}(v|n,x)$, $n = \underline{n}, \ldots, \overline{n}$:

$$\hat{F}(v|x) = \sum_{n=\underline{n}}^{\bar{n}} \hat{\pi}(n|x) \hat{F}(v|n,x). \tag{16}$$

3 Asymptotic properties

In this section, we discuss uniform consistency and asymptotic normality of the estimator of f proposed in the previous section. The consistency of the estimator of f follows from uniform consistency of its components. The following lemma establishes uniform convergence rates for the components of $\hat{f}$.

Lemma 1 *Let $\Lambda(x) = [v_1(x), v_2(x)] \subset [\underline{v}(x), \bar{v}(x)]$, $\Upsilon(x) = [\tau_1(x), \tau_2(x)]$, where $\tau_i(x) = F(v_i(x)|x)$ for $i = 1, 2$, and $\Theta(n, x) = [b_1(n, x), b_2(n, x)]$, where $b_i(n, x) = q(\tau_i(x)|n, x)$, $i = 1, 2$. Then, under Assumptions 1 and 2, for all $x \in Interior(\mathcal{X})$ and $n \in \mathcal{N}$,*

(a) $\hat{\pi}(n|x) - \pi(n|x) = O_p\left(\left(\frac{Lh^d}{\log L}\right)^{-1/2} + h^R\right)$.

(b) $\hat{\varphi}(x) - \varphi(x) = O_p\left(\left(\frac{Lh^d}{\log L}\right)^{-1/2} + h^R\right)$.

(c) $\sup_{b\in[\underline{b}(n,x),\bar{b}(n,x)]} |\hat{G}(b|n, x) - G(b|n, x)| = O_p\left(\left(\frac{Lh^d}{\log L}\right)^{-1/2} + h^R\right)$.

(d) $\sup_{\tau\in\Upsilon(x)} |\hat{q}(\tau|n, x) - q(\tau|n, x)| = O_p\left(\left(\frac{Lh^d}{\log L}\right)^{-1/2} + h^R\right)$.

(e) $\sup_{\tau\in\Upsilon(x)}(\lim_{t\downarrow\tau} \hat{q}(t|n, x) - \hat{q}(\tau|n, x)) = O_p\left(\left(\frac{Lh^d}{\log(Lh^d)}\right)^{-1}\right)$.

(f) $\sup_{b\in\Theta(n,x)} |\hat{g}^{(k)}(b|n, x) - g^{(k)}(b|n, x)| = O_p\left(\left(\frac{Lh^{d+1+2k}}{\log L}\right)^{-1/2} + h^R\right)$, $k = 0, \ldots, R$.

(g) $\sup_{\tau\in\Upsilon(x)} |\hat{Q}(\tau|n, x) - Q(\tau|x)| = O_p\left(\left(\frac{Lh^{d+1}}{\log L}\right)^{-1/2} + h^R\right)$.

(h) $\sup_{v\in\Lambda(x)} |\hat{F}(v|n, x) - F(v|x)| = O_p\left(\left(\frac{Lh^{d+1}}{\log L}\right)^{-1/2} + h^R\right)$.

As it follows from Lemma 1, the estimator of the derivative of $g(\cdot|n, x)$ has the slowest rate of convergence among all components of $\hat{f}$. Consequently, it determines the uniform convergence rate of $\hat{f}$.

Theorem 1 *Let $\Lambda(x)$ be as in Lemma 1. Then, under Assumptions 1 and 2, for all $x \in Interior(\mathcal{X})$, $\sup_{v\in\Lambda(x)} \left|\hat{f}(v|x) - f(v|x)\right| = O_p\left(\left(\frac{Lh^{d+3}}{\log L}\right)^{-1/2} + h^R\right)$.*

Remark. One of the implications of Theorem 1 is that our estimator achieves the optimal rate of GPV. Consider the following choice of the bandwidth parameter: $h = c\left(L/\log L\right)^{-\alpha}$. By choosing α so that $\left(Lh^{d+3}/\log L\right)^{-1/2}$ and h^R are of the same order, one obtains $\alpha = 1/\left(d+3+2R\right)$ and the rate $\left(L/\log L\right)^{-R/(d+3+2R)}$, which is the same as the optimal rate established in Theorem 2 of GPV.

Next, we discuss asymptotic normality of the proposed estimator. We make following assumption.

Assumption 3 *$Lh^{d+1} \to \infty$, and $\left(Lh^{d+1+2k}\right)^{1/2} h^R \to 0$.*

The rate of convergence and asymptotic variance of the estimator of f are determined by $\hat{g}^{(1)}\left(b|n,x\right)$, the component with the slowest rate of convergence. Hence, Assumption 3 will be imposed with $k=1$ which limits the possible choices of the bandwidth for kernel estimation. For example, if one follows the rule $h = cL^{-\alpha}$, then α has to be in the interval $\left(1/\left(d+3+2R\right), 1/\left(d+1\right)\right)$. As usual for asymptotic normality, there must be under smoothing relative to the optimal rate.

Lemma 2 *Let $\Theta\left(n,x\right)$ be as in Lemma 1. Then, under Assumptions 1-3,*

(a) $\left(Lh^{d+1+2k}\right)^{1/2}\left(\hat{g}^{(k)}\left(b|n,x\right) - g^{(k)}\left(b|n,x\right)\right) \to_d N\left(0, V_{g,k}\left(b,n,x\right)\right)$ *for* $b \in \Theta\left(n,x\right)$, $x \in Interior\left(\mathcal{X}\right)$, *and* $n \in \mathcal{N}$, *where*

$$V_{g,k}\left(b,n,x\right) = K_k g\left(b|n,x\right)/\left(n\pi\left(n|x\right)\varphi\left(x\right)\right),$$

and $K_k = \left(\int K^2\left(u\right)du\right)^d \int \left(K^{(k)}\left(u\right)\right)^2 du$.

(b) $\hat{g}^{(k)}\left(b|n_1,x\right)$ *and* $\hat{g}^{(k)}\left(b|n_2,x\right)$ *are asymptotically independent for all* $n_1 \neq n_2$, $n_1, n_2 \in \mathcal{N}$.

Now, we present the main result of the paper. By (48) in the Appendix, one obtains the following decomposition:

$$\begin{aligned}
&\hat{f}\left(v|n,x\right) - f\left(v|x\right) \\
= \ & \frac{F\left(v|x\right) f^2\left(v|x\right)}{\left(n-1\right) g^3\left(q\left(F\left(v|x\right)|n,x\right)|n,x\right)} \\
&\times\left(\hat{g}^{(1)}\left(q\left(F\left(v|x\right)|n,x\right)|n,x\right) - g^{(1)}\left(q\left(F\left(v|x\right)|n,x\right)|n,x\right)\right) \\
&+ o_p\left(\left(Lh^{d+3}\right)^{-1/2}\right).
\end{aligned} \tag{17}$$

Lemma 2, definition of $\hat{f}(v|x)$, and the decomposition in (17) lead to the following theorem.

Theorem 2 *Let $\Lambda(x)$ be as in Lemma 1. Then, under Assumptions 1, 2 and 3 with $k = 1$, and for $v \in \Lambda(x)$, $x \in Interior(\mathcal{X})$,*

$$\left(Lh^{d+3}\right)^{1/2}\left(\hat{f}(v|x) - f(v|x)\right) \to_d N(0, V_f(v, x)),$$

where $V_f(v, x)$ is given by

$$F^2(v|x) f^4(v|x) \sum_{n=\underline{n}}^{\overline{n}} \frac{\pi^2(n|x) V_{g,1}(q(F(v|x)|n, x), n, x)}{(n-1)^2 g^6(q(F(v|x)|n, x)|n, x)},$$

and $V_{g,1}(b, n, x)$ is defined in Lemma 2.

By Lemma 1, the asymptotic variance of $\hat{f}(v|x)$ can be consistently estimated by the plug-in estimator which replaces the unknown F, f, φ, π, g and q in the expression for $V_f(v, x)$ with their consistent estimators. In small samples, however, accuracy of the normal approximation can be improved by taking into the account the variance of the second-order term multiplied by h^2. To make the notation simple, consider the case of a *single* n. We can expand the decomposition in (17) to obtain that $\left(Lh^{d+3}\right)^{1/2}\left(\hat{f}(v|x, n) - f(v|x)\right)$ is given by

$$\frac{Ff^2}{(n-1)g^3}\left(Lh^{d+3}\right)^{1/2}\left(\hat{g}^{(1)} - g^{(1)}\right) + h\left(\frac{3f}{g} - \frac{2nf^2}{(n-1)g^2}\right)\left(Lh^d\right)^{1/2}(\hat{g} - g) + o_p(1),$$

where, F is the conditional CDF evaluated at v, and g, $g^{(1)}$, $\hat{g}$, $\hat{g}^{(1)}$ are the conditional density (given x and n), its derivative, and their estimators evaluated at $q(F(v|x)|n, x)$. With this decomposition, in practice, one can improve accuracy of asymptotic approximation by using the following expression for the estimated variance instead of $\hat{V}_f$ alone[7]:

$$\tilde{V}_f = \hat{V}_f + h^2\left(\frac{3\hat{f}}{\hat{g}} - \frac{2n\hat{f}^2}{(n-1)\hat{g}^2}\right)^2 \hat{V}_{g,0}.$$

[7]This is given that $\int K(u) K^{(1)}(u)\, du = 0$.

Note that the second summand in the expression for $\tilde{V}_f$ is $O_p(h^2)$ and negligible in large samples.

4 Inference on the optimal reserve price

In this section, we discuss inference on the optimal reserve price given x, $r^*(x)$. Riley and Samuelson (1981) show that under certain assumptions, $r^*(x)$ is given by the unique solution to the equation:

$$r^*(x) - \frac{1 - F(r^*(x)|x)}{f(r^*(x)|x)} - c = 0, \tag{18}$$

where c is the seller's own valuation. One approach to the inference on $r^*(x)$ is to estimate it as a solution $\hat{r}^*(x)$ to (18) using consistent estimators for f and F in place of the true unknown functions. However, a difficulty arises because, even though our estimator $\hat{f}(v|x)$ is asymptotically normal, it is not guaranteed to be a continuous function of v.

We instead take a direct approach to constructing CIs. We construct CIs for the optimal reserve price by inverting a collection of tests of the null hypotheses $r^*(x) = v$. The CIs are formed using all values v for which a test fails to rejects the null hypothesis that (18) holds at $r^*(x) = v$.[8]

Consider $H_0 : r^*(x) = v$, and a test statistic

$$T(v|x) = \left(Lh^{d+3}\right)^{1/2} \left(v - \frac{1 - \hat{F}(v|x)}{\hat{f}(v|x)} - c\right) / \sqrt{\frac{\left(1 - \hat{F}(v|x)\right)^2}{\hat{f}^4(v|x)} \hat{V}_f(v, x)},$$

where $\hat{F}$ is defined in (16), and $\hat{V}_f(v, x)$ is a consistent estimator of $V_f(v, x)$. By Theorem 2 and Lemma 1(h), $T(r^*(x)|x) \to_d N(0, 1)$. Furthermore, due to uniqueness of the solution to (18), for any $t > 0$, $P(|T(v|x)| > t | r^*(x) \neq v) \to 1$. A CI for r^* with the asymptotic coverage probability $1 - \alpha$ is formed by collecting all v's such

[8]Such CIs have been discussed in the econometrics literature, for example, in the presence of weak instruments (Andrews and Stock, 2005), for constructing CIs for the date of a structural break (Elliott and Müller, 2007), and inference on set identified parameters (Chernozhukov, Hong, and Tamer, 2004).

that a test based on $T(v|x)$ fails to reject the null at the significance level α:

$$CI_{1-\alpha}(x) = \left\{v : |T(v|x)| \leq z_{1-\alpha/2}\right\},$$

where z_τ is the τ quantile of the standard normal distribution. Note that such a CI asymptotically has correct coverage probability since by construction we have that $P(r^*(x) \in CI_{1-\alpha}(x)) = P\left(|T(r^*(x)|x)| \leq z_{1-\alpha/2}\right) \to 1-\alpha$.

5 Monte Carlo results

In this section, we evaluate small-sample accuracy of the asymptotic normal approximation for our estimator $\hat{f}(v)$ established in Theorem 2. We also compare small-sample properties of our estimator and the GPV estimator. We consider the case without covariates ($d=0$). The number of bidders, n, and the number of auctions, L, are chosen as follows: $n=5$, $L=500$, 5000, and 10000. The true distribution of valuations is chosen to be uniform over the interval $[0,3]$. We estimate f at the following points: $v = 0.8$, 1, 1.2, 1.4, 1.6, 1.8 and 2. Each Monte Carlo experiment has 1000 replications.

For each replication, we generate randomly nL valuations, $\{v_i : i = 1, \ldots, nL\}$, and then compute the corresponding bids according to the equilibrium bidding strategy $b_i = v_i(n-1)/n$. Computation of the quantile-based estimator $\hat{f}(v)$ involves several steps. First, we estimate $q(\tau)$, the quantile function of bids. Let $b_{(1)}, \ldots, b_{(nL)}$ denote the ordered sample of bids. We set $\hat{q}\left(\frac{i}{nL}\right) = b_{(i)}$. Second, we estimate $g(b)$, the PDF of bids using (8). Similarly to GPV, we use the triweight kernel with the bandwidth $h = 1.06\hat{\sigma}_b(nL)^{-1/5}$, where $\hat{\sigma}_b$ is the estimated standard deviation of bids. To construct our estimator, g needs to be estimated at all points $\left\{\hat{q}\left(\frac{i}{nL}\right) : i = 1, \ldots, nL\right\}$. In order to save on computation time, we estimate g at 120 equally spaced points on the interval $\left[\hat{q}\left(\frac{1}{nL}\right), \hat{q}(1)\right]$ and then interpolate to $\left\{\hat{q}\left(\frac{i}{nL}\right) : i = 1, \ldots, nL\right\}$ using the Matlab interpolation function `interp1`. Next, we compute $\left\{\hat{Q}^p\left(\frac{i}{nL}\right) : i = 1, \ldots, nL\right\}$ using (11), its monotone version according to (12), and $\hat{F}(v)$ according to (13). Let $\lceil x \rceil$ denote the nearest integer greater than or equal to x; we compute $\hat{q}\left(\hat{F}(v)\right)$ as $\hat{q}\left(\frac{\lceil nL\hat{F}(v)\rceil}{nL}\right)$. Next, we compute $\hat{g}\left(\hat{q}\left(\hat{F}(v)\right)\right)$ and $\hat{g}^{(1)}\left(\hat{q}\left(\hat{F}(v)\right)\right)$ using (8) and (9) respectively, and $\hat{f}(v)$ as the reciprocal of (15). Lastly, we compute the estimated asymptotic variance of $\hat{f}(v)$,

$$\hat{V}_f(v) = \frac{K_1 \hat{F}^2(v) \hat{f}^4(v)}{n(n-1)^2 \hat{g}^5\left(\hat{q}\left(\hat{F}(v)\right)\right)},$$

and the estimator of V_f that includes the variance of the second-order term:

$$\tilde{V}_f(v) = \hat{V}_f(v) + h^2 \left(\frac{3\hat{f}(v)}{\hat{g}\left(\hat{q}\left(\hat{F}(v)\right)\right)} - \frac{2n\hat{f}^2(v)}{(n-1)\hat{g}^2\left(\hat{q}\left(\hat{F}(v)\right)\right)} \right)^2 \hat{V}_{g,0}\left(\hat{q}\left(\hat{F}(v)\right)\right).$$

A CI with the asymptotic confidence level $1-\alpha$ is formed as

$$\hat{f}(v) \pm z_{1-\alpha/2}\sqrt{\hat{V}_f(v)/(Lh^3)} \text{ or } \hat{f}(v) \pm z_{1-\alpha/2}\sqrt{\tilde{V}_f(v)/(Lh^3)},$$

where z_τ is the τ quantile of the standard normal distribution.

Table 1 reports simulated coverage probabilities for 99%, 95% and 90% asymptotic CIs constructed using the first-order variance approximation $\hat{V}_f$. The results indicate that the first-order CIs tend to under cover, and the coverage probability error increases with v. This situation is observed in small ($L = 500$) and large samples ($L = 5000, 10000$) as well, and can be explained by the fact that $\hat{V}_f$ does not take into account variability associated with estimation of the higher-order terms. Table 2 reports coverage probabilities of the asymptotic CIs constructed using the corrected estimator of the variance, $\tilde{V}_f$. As the results indicate, the correction increased the estimated variance and brought the simulated coverage probabilities close to their nominal levels. The approximation appears to be more accurate for small values of v than for large. We conclude that the normal approximation using the corrected for second-order terms variance estimator provides a reasonably accurate description of the behavior of our estimator in finite samples.

Next, we compare the performance of our estimator with that of GPV. To compute the GPV estimator of $f(v)$, in the first step we compute nonparametric estimators of G and g, and obtain the pseudo-valuations $\hat{v}_{il}$ according to equation (1), with G and g replaced by their estimators. In the second step, we estimate $f(v)$ by the kernel method from the sample $\{\hat{v}_{il}\}$ obtained in the first-step. To avoid the boundary bias effect, GPV suggest trimming the observations that are too close to the estimated boundary of the support. Note that no explicit trimming is necessary for our estimator, since implicit trimming occurs from our use of quantiles instead of pseudo-valuations.

Our estimator can be expected to have worse small sample properties than GPV's, since it is a nonlinear function of the estimated PDF and its derivative, while the GPV estimator is obtained by kernel smoothing of the data on pseudo-valuations. Table 3 reports bias, mean-squared error (MSE), and median absolute deviation of the two estimators. The results show that except for a number of cases, the GPV estimator has smaller bias than the quantile-based estimator; however note that in very large samples ($L = 10000$) there are more cases in which the quantile-based estimator has a smaller bias. In all cases, the GPV's MSE and median absolute deviation are smaller than those of the quantile-based estimator. Furthermore, in the majority of cases, the ratio of the quantile-based MSE to the GPV MSE is remarkably close to 2.

Table 3 also reports the average (across replications) standard error of our estimator. As our theoretical results suggest, the variance of the estimator increases with v, since it depends on $F(v)$. This fact is also reflected in the MSE values that increase with v. Interestingly, one can see the same pattern for the MSE of the GPV estimator, which suggests that the GPV variance must be an increasing function of v as well.

6 Concluding remarks

In this note, we have developed a novel quantile-based estimator of the latent density of bidders' valuations $f(v)$ for first-price auctions. The estimator is shown to be consistent and asymptotically normal, and capable of converging at the optimal rate of GPV. We have compared the performance of both estimators in a limited Monte-Carlo study. We have found that the GPV estimator has smaller MSE and median absolute deviations than our estimator; however, in some cases our estimator has a smaller finite-sample bias. The emerging conclusion is that our approach is complementary to GPV. If one is interested in a relatively precise point estimate of $f(v)$, then the GPV estimator may be preferred, and especially so if the sample size is small. If, on the other hand, one is primarily interested in inferences about $f(v)$ rather than a point estimate, then our approach can provides a viable alternative, and especially so in moderately large samples.

Appendix of proofs

Proof of Lemma 1. Parts (a) and (b) of the lemma follow from Lemma B.3 of Newey (1994).

For part (c), define a function

$$G^0(b,n,x) = n\pi(n|x)\,G(b|n,x)\,\varphi(x),$$

and its estimator as

$$\hat{G}^0(b,n,x) = \frac{1}{L}\sum_{l=1}^{L}\sum_{i=1}^{n_l} 1(n_l = n)\,1(b_{il} \le b)\,K_{*h}(x_l - x). \tag{19}$$

Next,

$$\begin{aligned}
E\hat{G}^0(b,n,x) &= E\left(1(n_l = n)\,K_{*h}(x_l - x)\sum_{i=1}^{n_l} 1(b_{il} \le b)\right) \\
&= nE\left(1(n_l = n)\,1(b_{il} \le b)\,K_{*h}(x_l - x)\right) \\
&= nE\left(\pi(n|x_l)\,G(b|n,x_l)\,K_{*h}(x_l - x)\right) \\
&= n\int \pi(n|u)\,G(b|n,u)\,K_{*h}(u-x)\,\varphi(u)\,du \\
&= \int G^0(b,n,x+hu)\,K_d(u)\,du.
\end{aligned}$$

By Assumption 1(e) and Proposition 1(iii) of GPV, $G(b|n,\cdot)$ admits at least $R+1$ continuous bounded derivatives. Then, as in the proof of Lemma B.2 of Newey (1994), there exists a constant $c > 0$ such that

$$\begin{aligned}
&\left|G^0(b,n,x) - E\hat{G}^0(b,n,x)\right| \\
\le\ & ch^R\left(\int |K_d(u)|\,\|u\|^R\,du\right)\left\|vec\left(D_x^R G^0(b,n,x)\right)\right\|,
\end{aligned}$$

where $\|\cdot\|$ denotes the Euclidean norm, and $D_x^R G^0$ denotes the R-th partial derivative of G^0 with respect to x. It follows then that

$$\sup_{b\in\left[\underline{b}(n,x),\bar{b}(n,x)\right]}\left|G^0(b,n,x) - E\hat{G}^0(b,n,x)\right| = O\left(h^R\right). \tag{20}$$

Now, we show that

$$\sup_{b\in\left[\underline{b}(n,x),\bar{b}(n,x)\right]} |\hat{G}^0(b,n,x) - E\hat{G}^0(b,n,x)| = O_p\left(\left(\frac{Lh^d}{\log L}\right)^{-1/2}\right). \tag{21}$$

We follow the approach of Pollard (1984). Fix $n \in \mathcal{N}$ and $x \in Interior(\mathcal{X})$, and consider a class of functions $\mathcal{Z}$ indexed by h and b, with a representative function

$$z_l(b,n,x) = \sum_{i=1}^{n_l} 1(n_l = n)\, 1(b_{il} \leq b)\, h^d K_{*h}(x_l - x).$$

By the result in Pollard (1984) (Problem 28), the class $\mathcal{Z}$ has polynomial discrimination. Theorem 37 in Pollard (1984) (see also Example 38) implies that for any sequences δ_L, α_L such that $L\delta_L^2\alpha_L^2/\log L \to \infty$, $Ez_l^2(b,n,x) \leq \delta_L^2$,

$$\alpha_L^{-1}\delta_L^{-2} \sup_{b\in\left[\underline{b}(n,x),\bar{b}(n,x)\right]} |\frac{1}{L}\sum_{l=1}^{L} z_l(b,n,x) - Ez_l(b,n,x)| \to 0 \tag{22}$$

almost surely. We claim that this implies that

$$\sup_{b\in\left[\underline{b}(n,x),\bar{b}(n,x)\right]} |\hat{G}^0(b,n,x) - E\hat{G}^0(b,n,x)| = O_p\left(\left(\frac{Lh^d}{\log L}\right)^{-1/2}\right).$$

The proof is by contradiction. Suppose not. Then there exist a sequence $\gamma_L \to \infty$ and a subsequence of L such that along this subsequence,

$$\sup_{b\in\left[\underline{b}(n,x),\bar{b}(n,x)\right]} |\hat{G}^0(b,n,x) - E\hat{G}^0(b,n,x)| \geq \gamma_L\left(\frac{Lh^d}{\log L}\right)^{-1/2}. \tag{23}$$

on a set of events $\Omega' \subset \Omega$ with a positive probability measure. Now if we let $\delta_L^2 = h^d$ and $\alpha_L = (\frac{Lh^d}{\log L})^{-1/2}\gamma_L^{1/2}$, then the definition of z implies that, along the subsequence, on a set of events Ω',

$$\alpha_L^{-1}\delta_L^{-2} \sup_{b\in\left[\underline{b}(n,x),\bar{b}(n,x)\right]} |\frac{1}{L}\sum_{l=1}^{L} z_l(b,n,x) - Ez_l(b,n,x)|$$

$$
\begin{aligned}
&= \left(\frac{Lh^d}{\log L}\right)^{1/2} \gamma_L^{-1/2} h^{-d} \sup_{b\in\left[\underline{b}(n,x),\bar{b}(n,x)\right]} \left|\frac{1}{L}\sum_{l=1}^{L} z_l(b,n,x) - Ez_l(b,n,x)\right| \\
&= \left(\frac{Lh^d}{\log L}\right)^{1/2} \gamma_L^{-1/2} \sup_{b\in\left[\underline{b}(n,x),\bar{b}(n,x)\right]} \left|\hat{G}^0(b,n,x) - E\hat{G}^0(b,n,x)\right| \\
&\geq \left(\frac{Lh^d}{\log L}\right)^{1/2} \gamma_L^{-1/2} \gamma_L \left(\frac{Lh^d}{\log L}\right)^{-1/2} \\
&= \gamma_L^{1/2} \to \infty,
\end{aligned}
$$

where the inequality follows by (23), a contradiction to (22). This establishes (21), so that (20), (21) and the triangle inequality together imply that

$$
\sup_{b\in\left[\underline{b}(n,x),\bar{b}(n,x)\right]} \left|\hat{G}^0(b,n,x) - G^0(b,n,x)\right| = O_p\left(\left(\frac{Lh^d}{\log L}\right)^{-1/2} + h^R\right). \tag{24}
$$

To complete the proof, recall that, from the definitions of $G^0(b,n,x)$ and $\hat{G}^0(b,n,x)$,

$$
G(b|n,x) = \frac{G^0(b,n,x)}{\pi(n|x)\varphi(x)}, \text{ and } \hat{G}(b|n,x) = \frac{\hat{G}^0(b,n,x)}{\hat{\pi}(n|x)\hat{\varphi}(x)},
$$

so that by the mean-value theorem, $\left|\hat{G}(b|n,x) - G(b|n,x)\right|$ is bounded by

$$
\begin{aligned}
&\left\|\left(\frac{1}{\tilde{\pi}(n,x)\tilde{\varphi}(x)}, \frac{\tilde{G}^0(b,n,x)}{\tilde{\pi}^2(n,x)\tilde{\varphi}(x)}, \frac{\tilde{G}^0(b,n,x)}{\tilde{\pi}(n,x)\tilde{\varphi}^2(x)}\right)\right\| \\
&\times \left\|\left(\hat{G}^0(b,n,x) - G^0(b,n,x), \hat{\pi}(n|x) - \pi(n|x), \hat{\varphi}(x) - \varphi(x)\right)\right\|,
\end{aligned} \tag{25}
$$

where $\left\|\left(\tilde{G}^0 - G^0, \tilde{\pi} - \pi, \tilde{\varphi} - \varphi\right)\right\| \leq \left\|\left(\hat{G}^0 - G^0, \hat{\pi} - \pi, \hat{\varphi} - \varphi\right)\right\|$ for all (b,n,x). Further, by Assumption 1(b) and (c) and the results in parts (a) and (b) of the lemma, with the probability approaching one $\tilde{\pi}$ and $\tilde{\varphi}$ are bounded away from zero. The desired result follows from (24), (25) and parts (a) and (b) of the lemma.

For part (d) of the lemma, since $\hat{G}(\cdot|n,x)$ is monotone by construction,

$$
\begin{aligned}
P\left(\hat{q}(\tau_1(x)|n,x) < \underline{b}(n,x)\right) &= P\left(\inf_b\left\{b : \hat{G}(b|n,x) \geq \tau_1(x)\right\} < \underline{b}(n,x)\right) \\
&= P\left(\hat{G}\left(\underline{b}(n,x)|n,x\right) > \tau_1(x)\right)
\end{aligned}
$$

$$= \quad o(1),$$

where the last equality is by the result in part (c). Similarly,

$$\begin{aligned} P\left(\hat{q}(\tau_2(x)|n,x) > \bar{b}(n,x)\right) &= P\left(\hat{G}\left(\bar{b}(n,x)|n,x\right) < \tau_2(x)\right) \\ &= o(1). \end{aligned}$$

Hence, for all $x \in Interior(\mathcal{X})$ and $n \in \mathcal{N}$, with the probability approaching one, $\underline{b}(n,x) \leq \hat{q}(\tau_1(x)|n,x) < \hat{q}(\tau_2(x)|n,x) \leq \bar{b}(n,x)$. Since the distribution $G(b|n,x)$ is continuous in b,$G(q(\tau|n,x)|n,x) = \tau$, and, for $\tau \in \Upsilon(x)$, we can write the identity

$$G(\hat{q}(\tau|n,x)|n,x) - G(q(\tau|n,x)|n,x) = G(\hat{q}(\tau|n,x)|n,x) - \tau. \tag{26}$$

Using Lemma 21.1(ii) of van der Vaart (1998),

$$0 \leq \hat{G}(\hat{q}(\tau|n,x)|n,x) - \tau \leq \frac{1}{\hat{\pi}(n|x)\hat{\varphi}(x)nLh^d},$$

and by the results in (a) and (b),

$$\hat{G}(\hat{q}(\tau|n,x)|n,x) = \tau + O_p\left(\left(Lh^d\right)^{-1}\right) \tag{27}$$

uniformly over τ. Combining (26) and (27), and applying the mean-value theorem to the left-hand side of (26), we obtain

$$\begin{aligned} &\hat{q}(\tau|n,x) - q(\tau|n,x) \\ = \;& \frac{G(\hat{q}(\tau|n,x)|n,x) - \hat{G}(\hat{q}(\tau|n,x)|n,x)}{g(\widetilde{q}(\tau|n,x)|n,x)} + O_p\left(\left(Lh^d\right)^{-1}\right), \end{aligned} \tag{28}$$

where $\widetilde{q}$ lies between $\hat{q}$ and q for all (τ,n,x). Now, according to Proposition 1(ii) of GPV, there exists $c_g > 0$ such that $g(b|n,x) > c_g$ for all $b \in \left[\underline{b}(n,x), \bar{b}(n,x)\right]$, and the result in part (d) follows from (28) and part (c) of the lemma.

Next, we prove part (e) of the lemma. Fix $x \in Interior(\mathcal{X})$ and $n \in \mathcal{N}$. Let

$$N = \sum_{l=1}^{L}\sum_{i=1}^{n_l} 1(n_l = n) K_d(x_l).$$

Consider the ordered sample of bids $\underline{b}(n,x) = b_{(0)} \leq \ldots \leq b_{(N+1)} = \bar{b}(n,x)$ that corresponds to $n_l = n$ and $K_d(x_l) \neq 0$. Then,

$$0 \leq \lim_{t \downarrow \tau} \hat{q}(\tau|n,x) - \hat{q}(\tau|n,x) \leq \max_{j=1,\ldots,N+1} \left(b_{(j)} - b_{(j-1)}\right).$$

By the results of Deheuvels (1984),

$$\max_{j=1,\ldots,N+1} \left(b_{(j)} - b_{(j-1)}\right) = O_p\left(\left(\frac{N}{\log N}\right)^{-1}\right),$$

and part (e) follows, since $N = O_p\left(Lh^d\right)$.

To prove part (f), note that by Assumption 1(f) and Proposition 1(iv) of GPV, $g(\cdot|n,\cdot)$ admits at least $R+1$ continuous bounded partial derivatives. Let

$$g_0^{(k)}(b,n,x) = \pi(n|x)\, g^{(k)}(b|n,x)\, \varphi(x), \tag{29}$$

and define

$$\hat{g}_0^{(k)}(b,n,x) = \frac{1}{nL} \sum_{l=1}^{L} \sum_{i=1}^{n_l} 1(n_l = n)\, K_h^{(k)}(b_{il} - b)\, K_{*h}(x_l - x). \tag{30}$$

We can write the estimator $\hat{g}(b|n,x)$ as

$$\hat{g}(b|n,x) = \frac{\hat{g}_0(b,n,x)}{\hat{\pi}(n|x)\,\hat{\varphi}(x)},$$

so that

$$\hat{g}^{(k)}(b|n,x) = \frac{\hat{g}_0^{(k)}(b,n,x)}{\hat{\pi}(n|x)\,\hat{\varphi}(x)},$$

By Lemma B.3 of Newey (1994), $\hat{g}_0^{(k)}(b,n,x)$ is uniformly consistent over $b \in \Theta(n,x)$:

$$\sup_{b \in \Theta(n,x)} |\hat{g}_0^{(k)}(b,n,x) - g_0^{(k)}(b,n,x)| = O_p\left(\left(\frac{Lh^{d+1+2k}}{\log L}\right)^{-1/2} + h^R\right). \tag{31}$$

By the results in parts (a) and (b), the estimators $\hat{\pi}(n|x)$ and $\hat{\varphi}(x)$ converge at the rate faster than that in (31). The desired result follows by the same argument as in the proof of part (c), equation (25).

For part (g), let c_g be as in the proof of part (d) of the lemma. First, we consider the preliminary estimator, $\hat{Q}^p(\tau|n,x)$. We have that $\left|\hat{Q}^p(\tau|n,x) - Q(\tau|x)\right|$ is bounded by

$$
\begin{aligned}
& \left|\hat{q}(\tau|n,x) - q(\tau|n,x)\right| + \frac{\left|\hat{g}(\hat{q}(\tau|n,x)|n,x) - g(q(\tau|n,x)|n,x)\right|}{\hat{g}(\hat{q}(\tau|n,x)|n,x)\,c_g} \\
\leq\; & \left|\hat{q}(\tau|n,x) - q(\tau|n,x)\right| + \frac{\left|g(\hat{q}(\tau|n,x)|n,x) - g(q(\tau|n,x)|n,x)\right|}{\hat{g}(\hat{q}(\tau|n,x)|n,x)\,c_g} \\
& + \frac{\left|\hat{g}(\hat{q}(\tau|n,x)|n,x) - g(\hat{q}(\tau|n,x)|n,x)\right|}{\hat{g}(\hat{q}(\tau|n,x)|n,x)\,c_g} \\
\leq\; & \left(1 + \frac{\sup_{b\in\Theta(n,x)}\left|g^{(1)}(b|n,x)\right|}{\hat{g}(\hat{q}(\tau|n,x)|n,x)\,c_g}\right)\left|\hat{q}(\tau|n,x) - q(\tau|n,x)\right| \\
& + \frac{\left|\hat{g}(\hat{q}(\tau|n,x)|n,x) - g(\hat{q}(\tau|n,x)|n,x)\right|}{\hat{g}(\hat{q}(\tau|n,x)|n,x)\,c_g}. \qquad (32)
\end{aligned}
$$

Define $E_L(x) = \{\hat{q}(\tau_1(x)|n,x) \geq b_1(n,x), \hat{q}(\tau_2(x)|n,x) \leq b_2(n,x)\}$, and let $\beta_L = \left(\frac{Lh^{d+1+2k}}{\log L}\right)^{1/2} + h^{-R}$. By the result in part (d), $P(E_L^c(x)) = o(1)$. Hence, it follows from part (f) of the lemma the estimator $\hat{g}(\hat{q}(\tau|n,x)|n,x)$ is bounded away from zero with the probability approaching one. Consequently, it follows by Assumption 1(e) and part (d) of the lemma that the first summand on the right-hand side of (32) is $O_p\left(\beta_L^{-1}\right)$ uniformly over $\Upsilon(x)$. Next,

$$
\begin{aligned}
& P\left(\sup_{\tau\in\Upsilon(x)} \beta_L\left|\hat{g}(\hat{q}(\tau|n,x)|n,x) - g(\hat{q}(\tau|n,x)|n,x)\right| > M\right) \\
\leq\; & P\left(\sup_{\tau\in\Upsilon(x)} \beta_L\left|\hat{g}(\hat{q}(\tau|n,x)|n,x) - g(\hat{q}(\tau|n,x)|n,x)\right| > M, E_L(x)\right) \\
& + P\left(E_L^c(x)\right) \\
\leq\; & P\left(\sup_{b\in\Theta(x)} \beta_L\left|\hat{g}(b|n,x) - g(b|n,x)\right| > M\right) + o(1). \qquad (33)
\end{aligned}
$$

It follows from part (f) of the lemma and (33) that

$$
\sup_{\tau\in\Upsilon(x)} |\hat{Q}^p(\tau|n,x) - Q(\tau|x)| = O_p\left(\left(\frac{Lh^{d+1}}{\log L}\right)^{-1/2} + h^R\right). \qquad (34)
$$

Further, by construction, $\hat{Q}(\tau|n,x) - \hat{Q}^p(\tau|n,x) \geq 0$ for $\tau \geq \tau_0$. We can assume that $\tau_0 \in \Upsilon(x)$. Since $\hat{Q}^p(\cdot|n,x)$ is left-continuous, there exists $\tau' \in [\tau_0, \tau]$ such that $\hat{Q}^p(\tau'|n,x) = \sup_{t\in[\tau_0,\tau]} \hat{Q}^p(t|n,x)$. Since $Q(\cdot|x)$ is nondecreasing,

$$\begin{aligned}
& \hat{Q}(\tau|n,x) - \hat{Q}^p(\tau|n,x) \\
= \ & \hat{Q}^p(\tau'|n,x) - \hat{Q}^p(\tau|n,x) \\
\leq \ & \hat{Q}^p(\tau'|n,x) - Q(\tau'|x) + Q(\tau|x) - \hat{Q}^p(\tau|n,x) \\
\leq \ & \sup_{t\in[\tau_0,\tau]} \left(\hat{Q}^p(t|n,x) - Q(t|x)\right) + Q(\tau|x) - \hat{Q}^p(\tau|n,x) \\
\leq \ & 2 \sup_{\tau\in\Upsilon(x)} \left|\hat{Q}^p(\tau|n,x) - Q(\tau|x)\right| \\
= \ & O_p\left(\left(\frac{Lh^{d+1}}{\log L}\right)^{-1/2} + h^R\right),
\end{aligned}$$

where the last result follows from (34). Using a similar argument for $\tau < \tau_0$, we conclude that

$$\sup_{\tau\in\Upsilon(x)} \left|\hat{Q}(\tau|n,x) - \hat{Q}^p(\tau|x)\right| = O_p\left(\left(\frac{Lh^{d+1}}{\log L}\right)^{-1/2} + h^R\right). \tag{35}$$

The result of part (g) follows from (34) and (35).

Lastly, we prove part (h). By construction $\hat{F}(\cdot|n,x)$ is a nondecreasing function.

$$\begin{aligned}
& P\left(\hat{F}(Q(\tau_1(x)|x)|n,x) < \tau_1(x)\right) \\
= \ & P\left(\sup_t \left\{t : \hat{Q}(t|n,x) \leq Q(\tau_1(x)|x)\right\} < \tau_1(x)\right) \\
\leq \ & P\left(\hat{Q}(\tau_1(x)|n,x) > Q(\tau_1(x)|x)\right) \\
= \ & o(1),
\end{aligned}$$

where the last equality follows from part (f) of the lemma. Further, due to monotonicity of $\hat{Q}(\cdot|n,x)$,

$$\begin{aligned}
& P\left(\hat{F}(Q(\tau_1(x)|x)|n,x) > \tau_2(x)\right) \\
= \ & P\left(\sup_t \left\{t : \hat{Q}(t|n,x) \leq Q(\tau_1(x)|x)\right\} > \tau_2(x)\right) \\
\leq \ & P\left(\hat{Q}(\tau_2(x)|n,x) < Q(\tau_1(x)|x)\right)
\end{aligned}$$

$$= \quad o(1).$$

By a similar argument one can establish that $P\left(\hat{F}(Q(\tau_2(x)|x)|n,x) \in \Upsilon(x)\right)$ converges to one, and, therefore, for all $v \in \Lambda(x)$, $\hat{F}(v|n,x) \in \Upsilon(x)$ with the probability approaching one. Next, by Assumption 1(f), $F(\cdot|x)$ is continuously differentiable on $\Lambda(x)$ and, therefore, $Q(\cdot|x)$ is continuously differentiable on $\Upsilon(x)$. By the mean-value theorem we have that for all $v \in \Lambda(x)$ with the probability approaching one,

$$\begin{aligned} Q\left(\hat{F}(v|n,x)|x\right) - v &= Q\left(\hat{F}(v|n,x)|x\right) - Q(F(v|x)) \\ &= \frac{1}{f\left(\widetilde{F}(v|n,x)|x\right)}\left(\hat{F}(v|n,x) - F(v|x)\right). \end{aligned} \tag{36}$$

where $\widetilde{F}(v|n,x)$ is in between $\hat{F}(v|n,x)$ and $F(v|x)$.

Similarly to Lemma 21.1(ii) of van der Vaart (1998), $\hat{Q}\left(\hat{F}(v|n,x)|n,x\right) \le v$, and equality can fail only at the points of discontinuity of $\hat{Q}$. Hence,

$$\begin{aligned} &\sup_{v\in\Lambda(x)}\left(v - \hat{Q}\left(\hat{F}(v|n,x)|n,x\right)\right) \\ \le\ &\sup_{\tau\in\Upsilon(x)}\left(\lim_{t\downarrow\tau}\hat{Q}(t|n,x) - \hat{Q}(\tau|n,x)\right) \\ \le\ &\left(1 + \frac{\sup_{b\in\Theta(n,x)}\left|\hat{g}^{(1)}(b|n,x)\right|}{\hat{g}^2(\hat{q}(\tau|n,x)|n,x)}\right)\sup_{\tau\in\Upsilon(x)}\left(\lim_{t\downarrow\tau}\hat{q}(t|n,x) - \hat{q}(\tau|n,x)\right) \\ =\ &O_p\left(\left(\frac{Lh^d}{\log(Lh^d)}\right)^{-1}\right), \end{aligned} \tag{37}$$

where the second inequality follows from the definition of $\hat{Q}$ and by continuity of K, and the equality (37) follows from part (e) of the lemma. Combining (36) and (37), and by Assumption 1(e) we obtain that there exists a constant $c > 0$ such that $\sup_{v\in\Lambda(x)}\left|\hat{F}(v|n,x) - F(v|x)\right|$ is bounded by

$$\begin{aligned} &c\sup_{v\in\Lambda(x)}\left|Q\left(\hat{F}(v|n,x)|x\right) - \hat{Q}\left(\hat{F}(v|n,x)|n,x\right)\right| + O_p\left(\left(\frac{Lh^d}{\log(Lh^d)}\right)^{-1}\right) \\ \le\ &c\sup_{\tau\in\Upsilon(x)}\left|Q(\tau|x) - \hat{Q}(\tau|n,x)\right| + O_p\left(\left(\frac{Lh^d}{\log(Lh^d)}\right)^{-1}\right) \end{aligned}$$

$$= O_p\left(\left(\frac{Lh^{d+1}}{\log L}\right)^{-1/2} + h^R\right),$$

where the equality follows from part (g) of the lemma. ■

Proof of Theorem 1. By Lemma 1(d),(f) and (h), $\hat{q}\left(\hat{F}(v|n,x)|n,x\right) \in \Theta(n,x)$ with the probability approaching one. Next,

$$\begin{aligned}
& \left|\hat{g}^{(1)}\left(\hat{q}\left(\hat{F}(v|n,x)|n,x\right)|n,x\right) - g^{(1)}\left(q\left(F(v|x)|n,x\right)|n,x\right)\right| \\
\leq & \sup_{b\in\Theta(n,x)}\left|\hat{g}^{(1)}(b|n,x) - g^{(1)}(b|n,x)\right| \\
& + g^{(2)}\left(\widetilde{q}(v,n,x)\right)\left|\hat{q}\left(\hat{F}(v|n,x)|n,x\right) - q\left(F(v|x)|n,x\right)\right|. \qquad (38)
\end{aligned}$$

where $\widetilde{q}$ is the mean value between $\hat{q}$ and q. Further, $g^{(2)}$ is bounded by Assumption 1(e) and Proposition 1(iv) of GPV, and

$$\begin{aligned}
& \left|\hat{q}\left(\hat{F}(v|n,x)|n,x\right) - q\left(F(v|x)|n,x\right)\right| \\
\leq & \sup_{\tau\in\Upsilon(x)}\left|\hat{q}(\tau|n,x) - q(\tau|n,x)\right| + \frac{1}{c_g}\sup_{v\in\Lambda(x)}\left|\hat{F}(v|n,x) - F(v|x)\right|, \qquad (39)
\end{aligned}$$

where c_g as in the proof of Lemma 1(d). By (38), (39) and Lemma 1(d),(f),(h),

$$\begin{aligned}
& \sup_{v\in\Lambda(x)}\left|\hat{g}^{(1)}\left(\hat{q}\left(\hat{F}(v|n,x)|n,x\right)|n,x\right) - g^{(1)}\left(q\left(F(v|x)|n,x\right)|n,x\right)\right| \\
= & O_p\left(\left(\frac{Lh^{d+3}}{\log L}\right)^{-1/2} + h^R\right). \qquad (40)
\end{aligned}$$

By a similar argument,

$$\begin{aligned}
& \hat{f}(v|n,x) - f(v|n,x) \\
= & \frac{F(v|x)\widetilde{f}^2(v|n,x)}{(n-1)g^3\left(q\left(F(v|x)|n,x\right)|n,x\right)} \\
& \times\left|\hat{g}^{(1)}\left(\hat{q}\left(\hat{F}(v|n,x)|n,x\right)|n,x\right) - g^{(1)}\left(q\left(F(v|x)|n,x\right)|n,x\right)\right| \\
& + O_p\left(\left(\frac{Lh^{d+1}}{\log L}\right)^{-1/2} + h^R\right), \qquad (41)
\end{aligned}$$

uniformly in $v \in \Lambda(x)$, where $\widetilde{f}(v|x)$ as in (15) but with some mean value $\widetilde{g}^{(1)}$ between

$g^{(1)}$ and its estimator $\hat{g}^{(1)}$. The desired result follows from (14), (40) , (41) and Lemma 1(a). ■

Proof of Lemma 2. Consider $g_0^{(k)}(b,n,x)$ and $\hat{g}_0^{(k)}(b,n,x)$ defined in (29) and (30) respectively. It follows from parts (a) and (b) of Lemma 1,

$$\begin{aligned} & \left(Lh^{d+1+2k}\right)^{1/2}\left(\hat{g}^{(k)}\left(b|n,x\right)-g^{(k)}\left(b|n,x\right)\right) \\ = & \frac{1}{\pi\left(n|x\right)\varphi\left(x\right)}\left(Lh^{d+1+2k}\right)^{1/2}\left(\hat{g}_0^{(k)}\left(b,n,x\right)-g_0^{(k)}\left(b,n,x\right)\right)+o_p(1). \end{aligned} \tag{42}$$

By the same argument as in the proof of part (f) of Lemma 1 and Lemma B2 of Newey (1994), $E\hat{g}_0^{(k)}\left(b,n,x\right)-g_0^{(k)}\left(b,n,x\right)=O\left(h^R\right)$ uniformly in $b\in\Theta\left(n,x\right)$ for all $x\in Interior\left(\mathcal{X}\right)$ and $n\in\mathcal{N}$. Then, by Assumption 3, it remains to establish asymptotic normality of

$$\left(nLh^{d+1+2k}\right)^{1/2}\left(\hat{g}_0^{(k)}\left(b,n,x\right)-E\hat{g}_0^{(k)}\left(b,n,x\right)\right).$$

Define

$$\begin{aligned} w_{il,n} &= h^{(d+1+2k)/2}1\left(n_l=n\right)K_h^{(k)}\left(b_{il}-b\right)K_{*h}\left(x_l-x\right), \\ \overline{w}_{L,n} &= (nL)^{-1}\sum_{l=1}^{L}\sum_{i=l}^{n_l}w_{il,n}, \end{aligned}$$

so that

$$\begin{aligned} & \left(nLh^{d+1+2k}\right)^{1/2}\left(\hat{g}_0^{(k)}\left(b,n,x\right)-E\hat{g}_0^{(k)}\left(b,n,x\right)\right) \\ = & (nL)^{1/2}\left(\overline{w}_{L,n}-E\overline{w}_{L,n}\right). \end{aligned} \tag{43}$$

By the Liapunov CLT (see, for example, Corollary 11.2.1 on page 427 of Lehman and Romano (2005)),

$$(nL)^{1/2}\left(\overline{w}_{L,n}-E\overline{w}_{L,n}\right)/\left(nLVar\left(\overline{w}_{L,n}\right)\right)^{1/2}\to_d N\left(0,1\right), \tag{44}$$

provided that $Ew_{il,n}^2<\infty$, and for some $\delta>0$,

$$\lim_{L\to\infty}\frac{1}{L^{\delta/2}}E\left|w_{il,n}\right|^{2+\delta}=0. \tag{45}$$

The condition in (45) follows from the Liapunov's condition (equation (11.12) on page 427 of Lehman and Romano (2005)), c_r inequality and because $w_{il,n}$ are i.i.d. Next, $Ew_{il,n}$ is given by

$$\begin{aligned}
& h^{(d+1+2k)/2}E\left(\pi\left(n|x_l\right)\int K_h^{(k)}\left(u-b\right)g\left(u|n,x_l\right)duK_{*h}\left(x_l-x\right)\right) \\
= \; & h^{(d+1+2k)/2}\int \pi\left(n|y\right)K_{*h}\left(y-x\right)\varphi\left(y\right)\int K_h^{(k)}\left(u-b\right)g\left(u|n,y\right)dudy \\
= \; & h^{(d+1)/2}\int \pi\left(n|hy+x\right)K_d\left(y\right)\varphi\left(hy+x\right) \\
& \times\int K^{(k)}\left(u\right)g\left(hu+b|n,hy+x\right)dudy \\
\to \; & 0.
\end{aligned}$$

Further, $Ew_{il,n}^2$ is given by

$$\begin{aligned}
& h^{d+1+2k}\int \pi\left(n|y\right)K_{*h}^2\left(y-x\right)\varphi\left(y\right)\int\left(K_h^{(k)}\left(u-b\right)\right)^2 g\left(u|n,y\right)dudy \\
= \; & \int \pi\left(n|hy+x\right)K_d^2\left(y\right)\varphi\left(hy+x\right) \\
& \times\int\left(K^{(k)}\left(u\right)\right)^2 g\left(hu+b|n,hy+x\right)dudy.
\end{aligned}$$

Hence, $nLVar\left(\overline{w}_{L,n}\right)$ converges to

$$\pi\left(n|x\right)g\left(b|n,x\right)\varphi\left(x\right)\left(\int K^2\left(u\right)du\right)^d\int\left(K^{(k)}\left(u\right)\right)^2du. \tag{46}$$

Lastly, $E\left|w_{il,n}\right|^{2+\delta}$ is given by

$$\begin{aligned}
& h^{(d+1+2k)(1+\delta/2)} \\
& \times\int \pi\left(n|y\right)\left|K_{*h}\left(y-x\right)\right|^{2+\delta}\varphi\left(y\right)\int\left|K_h^{(k)}\left(u-b\right)\right|^{2+\delta}g\left(u|n,y\right)dudy \\
= \; & h^{-(d+1)\delta/2}\int \pi\left(n|hy+x\right)\left|K_d\left(y\right)\right|^{2+\delta}\varphi\left(hy+x\right) \\
& \times\int\left|K^{(k)}\left(u\right)\right|^{2+\delta}g\left(hu+b|n,hy+x\right)dudy \\
\leq \; & h^{-(d+1)\delta/2}c_g\sup_{u\in[-1,1]}\left|K\left(u\right)\right|^{d(2+\delta)}\sup_{x\in\mathcal{X}}\varphi\left(x\right)\sup_{u\in[-1,1]}\left|K^{(k)}\left(u\right)\right|^{2+\delta},
\end{aligned} \tag{47}$$

where c_g as in the proof of Lemma 1(d). The condition (45) is satisfied by Assumptions 1(b) and 3, and (47). It follows now from (42)-(47),

$$\begin{aligned}
&\left(nLh^{d+3}\right)^{1/2}\left(\hat{g}^{(k)}\left(b|n,x\right)-g^{(k)}\left(b|n,x\right)\right)\\
&\to_d N\left(0,\frac{g\left(b|n,x\right)}{\pi\left(n|x\right)\varphi\left(x\right)}\left(\int K^2\left(u\right)du\right)^d\int\left(K^{(k)}\left(u\right)\right)^2du\right).
\end{aligned}$$

To prove part (b), note that the asymptotic covariance of $\overline{w}_{L,n_1}$ and $\overline{w}_{L,n_2}$ involves the product of two indicator functions, $1\left(n_l=n_1\right)1\left(n_l=n_2\right)$, which is zero for $n_1\neq n_2$. The joint asymptotic normality and asymptotic independence of $\hat{g}^{(k)}\left(b|n_1,x\right)$ and $\hat{g}^{(k)}\left(b|n_2,x\right)$ follows then by the Cramér-Wold device. ■

Proof of Theorem 2. First,

$$\begin{aligned}
&\hat{g}^{(1)}\left(\hat{q}\left(\hat{F}\left(v|n,x\right)|n,x\right)|n,x\right)-g^{(1)}\left(q\left(F\left(v|x\right)|n,x\right)|n,x\right)\\
=\;&\hat{g}^{(1)}\left(q\left(F\left(v|x\right)|n,x\right)|n,x\right)-g^{(1)}\left(q\left(F\left(v|x\right)|n,x\right)|n,x\right)\\
&+\hat{g}^{(2)}\left(\widetilde{q}\left(v,n,x\right)|n,x\right)\left(\hat{q}\left(\hat{F}\left(v|n,x\right)|n,x\right)-q\left(F\left(v|x\right)|n,x\right)\right),
\end{aligned}\tag{48}$$

where $\widetilde{q}$ is the mean value. It follows from Lemma 1(d) and (f) that the second summand on the right-hand side of the above equation is $o_p\left(\left(Lh^{d+3}\right)^{-1/2}\right)$. One arrives at (17), and the desired result follows immediately from (14), (17), Theorem 1, and Lemma 2. ■

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Table 1: Simulated coverage probabilities of CIs (constructed using the first-order approximation of the variance) for different valuations (v), numbers of auctions (L), and the Uniform (0,3) distribution

	v						
nominal confidence level	0.8	1.0	1.2	1.4	1.6	1.8	2.0
				$L = 500$			
0.99	0.964	0.952	0.942	0.947	0.944	0.926	0.925
0.95	0.909	0.913	0.892	0.894	0.898	0.874	0.869
0.90	0.847	0.864	0.848	0.848	0.854	0.842	0.827
				$L = 5000$			
0.99	0.980	0.977	0.977	0.971	0.974	0.965	0.958
0.95	0.922	0.927	0.931	0.926	0.936	0.931	0.916
0.90	0.879	0.885	0.877	0.890	0.894	0.894	0.882
				$L = 10000$			
0.99	0.975	0.978	0.973	0.977	0.979	0.977	0.960
0.95	0.923	0.931	0.930	0.932	0.938	0.929	0.923
0.90	0.866	0.886	0.884	0.894	0.907	0.890	0.887

Table 2: Simulated coverage probabilities of CIs (constructed using the second-order approximation of the variance) for different valuations (v), numbers of auctions (L), and the Uniform (0,3) distribution

	v						
nominal confidence level	0.8	1.0	1.2	1.4	1.6	1.8	2.0
				$L = 500$			
0.99	0.985	0.985	0.980	0.975	0.972	0.964	0.949
0.95	0.963	0.949	0.925	0.935	0.928	0.899	0.900
0.90	0.916	0.911	0.892	0.891	0.888	0.865	0.857
				$L = 5000$			
0.99	0.989	0.987	0.987	0.974	0.980	0.970	0.966
0.95	0.950	0.940	0.946	0.937	0.945	0.936	0.923
0.90	0.899	0.895	0.892	0.900	0.900	0.901	0.890
				$L = 10000$			
0.99	0.985	0.982	0.982	0.985	0.980	0.979	0.964
0.95	0.941	0.939	0.938	0.935	0.944	0.942	0.930
0.90	0.893	0.896	0.893	0.902	0.913	0.898	0.893

Table 3: Bias, MSE and median absolute deviation of the quantile-based (QB) and GPV estimators, and the average standard error (corrected) of the QB estimator for different valuations (v), numbers of auctions (L) and the Uniform (0,3) distribution

	Bias		MSE		Med abs deviation		Std error
v	QB	GPV	QB	GPV	QB	GPV	QB
				$L = 500$			
0.8	**-0.0011**	**-0.0011**	0.0020	0.0012	0.0305	0.0235	0.0463
1.0	0.0033	0.0018	0.0034	0.0019	0.0375	0.0306	0.0567
1.2	**-0.0002**	**-0.0004**	0.0043	0.0023	0.0439	0.0337	0.0657
1.4	0.0010	0.0005	0.0067	0.0029	0.0470	0.0358	0.0778
1.6	-0.0014	-0.0012	0.0072	0.0033	0.0493	0.0373	0.0864
1.8	-0.0046	0.0016	0.0107	0.0043	0.0575	0.0442	0.0981
2.0	0.0066	0.0009	0.0220	0.0052	0.0653	0.0494	0.1262
				$L = 5000$			
0.8	0.0002	0.0000	0.0007	0.0004	0.0177	0.0131	0.0266
1.0	**-0.0006**	**-0.0006**	0.0010	0.0005	0.0217	0.0166	0.0325
1.2	-0.0007	0.0001	0.0015	0.0008	0.0261	0.0198	0.0386
1.4	-0.0024	-0.0018	0.0019	0.0010	0.0290	0.0215	0.0446
1.6	0.0020	0.0016	0.0027	0.0013	0.0338	0.0247	0.0521
1.8	0.0013	0.0000	0.0035	0.0016	0.0357	0.0264	0.0587
2.0	0.0035	0.0028	0.0041	0.0019	0.0408	0.0290	0.0661
				$L = 10000$			
0.8	0.0018	0.0012	0.0006	0.0003	0.0156	0.0119	0.0230
1.0	**0.0001**	**-0.0002**	0.0008	0.0004	0.0195	0.0135	0.0280
1.2	**0.0004**	**0.0005**	0.0011	0.0005	0.0222	0.0160	0.0335
1.4	-0.0015	-0.0014	0.0013	0.0006	0.0250	0.0180	0.0385
1.6	0.0031	0.0024	0.0021	0.0010	0.0293	0.0211	0.0452
1.8	**-0.0011**	**-0.0014**	0.0024	0.0011	0.0321	0.0228	0.0497
2.0	0.0024	0.0018	0.0033	0.0014	0.0356	0.0245	0.0566